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On discontinuous strain fields in incompressible finite elastostatics

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Abstract

Piece-wise homogeneous three-dimensional deformations in incompressible materials in finite elasticity are considered. The emergence of discontinuous strain fields in incompressible materials is studied via singularity theory. Since the simplest singularities, including Maxwell's sets, are the cusp singularities, cusp conditions for the total energy function of homogeneous deformations for incompressible materials in finite elasticity will be derived, compatible with strain jumping. The proposed method yields simple criteria for the study of discontinuous deformations in three-dimensional problems and for any homogeneous incompressible material. Furthermore the homogeneous stress tensor is also not restricted. Neither fictitious nor simplified constitutive relations are invoked. The theory is implemented in a simple shearing problem.

Keywords: Finite elasticity; Continuum mechanics; Singularities; Discontinous strain; Bifurcations; Two phase states

1. Introduction

The author (Lazopoulos, in press) has recently presented a paper concerning the emergence of discontinuous strain fields in finite elastostatics. The method is based upon singularity theory. The present work extends the proposed method to the emergence of discontinuous deformations in incompressible materials.

Ericksen (1975) introduced two-phase deformations in solid materials, adopting globally stable equilibrium states in the class of deformations with smooth displacements but with non-smooth elastic strain fields. Various kinds of two-phase deformations cover austenite–martensite transformations (Khatchaturyan, 1983), and ferroelastic materials (Salze, 1990), the twinning phenomena in crystals (Pitteri and

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Zanzotto, 2003). Truskivovsky and Zanzotto (1996) proposed multiphase equilibrium problems but with continuous strain fields, employing terms of strain gradient into the strain energy density function. Recent work concerning the emergence of discontinuous strain fields in incompressible materials may be found in De Tommasi et al. (2001) and D'Ambrosio et al. (2003). Further references are included in Lazopoulos (in press).

According to Lazopoulos (in press), Maxwell's sets yielding multiple global minima may be found in the neighborhoods of singularities for the total potential energy function higher or equal to the cusp. Lazopoulos (submitted for publication) has also proposed a method for the classification of the singularities of the potential energy function under homogeneous deformation and the action of multiple constraints, see also Lazopoulos and Markatis (1994). Introducing the jumping of the strain compatibility condition (James, 1981; Gurtin, 1983), the simple singularities (cuspoids) of total potential energy function are classified and the cusps are located. The global minima of the total energy function are completely defined with the help of Maxwell's set of the cusp, describing the discontinuous strain fields with their phase boundary. No specific strain energy density functions are needed. The method does not restrict the constant stress tensor as far as cusp singularities of the total energy density may be probed.

The present problem of the emergence of discontinuous strain fields in incompressible elastic materials considers smooth stress–strain relations and the non-convexities of the potential energy function, under the incompressibility constraint, are not pre-assigned, as it is usual in these problems, but probed through singularity theory tools. The present procedure may be extended to materials with multiple constraints. Further no isotropy restrictions are imposed.

The theory is implemented in a simple shearing problem of Blatz and Ko (1962) material explaining the various steps. Mathematica computerized algebra package (Wolfram, 1996), has been applied for deriving the various formulae and computing.

2. The 1-D discontinuous strain problem

Ericksen (1975) introduced the coexistence of phases phenomena in solids discussing the globally stable equilibrium states in a tensioned bar with non-convex strain energy density. When the stress reaches Maxwell's value, continuous strain fields with discontinuous derivatives (strain) may emerge. Just the same result may be obtained with the help of Erdman–Weierstrass corner conditions (Gelfand and Fomin, 1963), of the variational problem concerning a tensioned bar with non-convex strain energy density. Details may be found in Lazopoulos (in press). Furthermore, the author introduces a different procedure for the study of the non-smooth stain fields, using singularity theory. In fact, if the bar is deformed by tension stress σ and the displacement is expressed by $u(x)$, the strain $u'(x)$ initially is constant. Let us assume that the stress σ_0 yields constant strain u'_0. Increasing the stress by $\mathrm{d}\sigma$ the incremental constant strain is $\mathrm{d}u'(x)$. If the strain energy density of the bar is given by a cusp unfolding, i.e. by

$$W(u'(x)) = \mathrm{d}u'^4 - a_0\mathrm{d}u'^2 + b_0\mathrm{d}u' + W(u'_0) \tag{1}$$

with $a_0, b_0 > 0$, Maxwell's sets, generating equal minima, are defined if the incremental stress $\mathrm{d}\sigma_0 = b_0$. In this case

$$\mathrm{d}u' = \pm\sqrt{\frac{2a_0}{3}}. \tag{2}$$

Hence the strain is equal to

$$u'(x) = u'_0 \pm \sqrt{\frac{2a_0}{3}}. \tag{3}$$

Hence, there exist regions of piece-wise constant distributed strain in the bar. Sometimes the distribution is so fine, that it seems to be like another homogeneous deformation.

3. Mathematical formulation

Following the same formulation as in Lazopoulos (in press), we consider an anisotropic homogeneous cube in its unstressed placement. The total energy function per unit volume of the stressed cube is

$$V = W(e_{ij}) - t_{ij}u_{ij}, \tag{4}$$

where t_{ij} is the first Piola–Kirchhoff stress referred to the unstressed reference placement, e_{ij} is the non-linear Green strain tensor (Ogden, 1997) and u_{ij} the displacement gradient tensor. As it is explained in Lazopoulos (in press), the number independent displacement gradient elements is reduced from nine to six, due to the conservation of rotational momentum. Therefore, the total potential energy function

$$V = V(v_i, t_k) = V(u_{ab}, t_{cd}) \quad i = 1, \ldots, 6, \ k = 1, \ldots, 9 \quad \text{and} \quad a, b, c, d = 1, \ldots, 3, \tag{5}$$

where

$$v_1 = u_{11}, \quad v_2 = u_{22}, \quad v_3 = u_{33}, \quad v_4 = u_{12}, \quad v_5 = u_{13}, \quad v_6 = u_{23} \tag{6}$$

and

$$t_1 = t_{11}, \quad t_2 = t_{22}, \quad t_3 = t_{33}, \quad t_4 = t_{12}, \quad t_5 = t_{13}, \quad t_6 = t_{23}, \quad t_7 = t_{21}, \quad t_8 = t_{31}, \quad t_9 = t_{32}. \tag{7}$$

Hence, the total energy function is defined in the $R^6 \times R^9$ space. Since the material is incompressible, the v_i, $i = 1, \ldots, 6$ do not vary independently, but they are subjected to the constraint,

$$J(v_i) = \det(F) = \det(\delta_{ij} + u_{ij}) = 1. \tag{8}$$

It is evident the values of the entries of the matrix **A** depend on the equilibrium point. Studying, further, the equilibrium of the system in the generalized six-dimensional vector space v_i at the point v_i^0 under the action of the nine-dimensional control parameters t_j^0, we denote by

$$\mathbf{y} = (v_i, t_j) \tag{9}$$

the 15-dimensional space of the generalized vectors v_i and the control stress vectors t_j. Let us consider

$$\mathbf{y^0} = (v_i^0, t_j^0) \tag{10}$$

is an equilibrium point. The problem now is posed as follows:

Perturbing the parameters t_j^0 so that

$$t_j = t_j^0 + \mathrm{d}t_j \tag{11}$$

with $|\mathrm{d}t_j| \ll 1$, find the new equilibrium generalized displacement gradient vector,

$$v_i = v_i^0 + \mathrm{d}v_i \tag{12}$$

in the neighborhood of the equilibrium placement defined by the displacement gradient v_i^0.

Let us point out that unique incremental displacement vector $\mathrm{d}v_i$ characterizes the stable elastic systems. However, the critical states are distinguished by the multiple incremental generalized displacement vector $\mathrm{d}v_i$. When none $\mathrm{d}v_i$ could be found, under the action of the incremental stress parameter $\mathrm{d}t_j$, equilibrium breaks down and motion of the system is expected.

Following Lazopoulos (submitted for publication) and Lazopoulos and Markatis (1994) the equilibrium equation is defined by the equation

$$\mathbf{A}\mathbf{d}\mathbf{v} = \mathbf{0}, \tag{13}$$

where $\mathbf{A}$ is the 2×6 matrix,

$$\mathbf{A} = \begin{bmatrix} \frac{\partial V}{\partial v_1} & \frac{\partial V}{\partial v_2} & \frac{\partial V}{\partial v_3} & \frac{\partial V}{\partial v_4} & \frac{\partial V}{\partial v_5} & \frac{\partial V}{\partial v_6} \\ \frac{\partial J}{\partial v_1} & \frac{\partial J}{\partial v_2} & \frac{\partial J}{\partial v_3} & \frac{\partial J}{\partial v_4} & \frac{\partial J}{\partial v_5} & \frac{\partial J}{\partial v_6} \end{bmatrix} \tag{14}$$

and

$$\mathbf{dv} = \begin{bmatrix} \mathrm{d}v_1 \\ \mathrm{d}v_2 \\ \mathrm{d}v_3 \\ \mathrm{d}v_4 \\ \mathrm{d}v_5 \\ \mathrm{d}v_6 \end{bmatrix}. \tag{15}$$

Hence, Eq. (13) should be satisfied by five independent solutions $\mathbf{dv}$. The space of those solutions is denoted by a 6×5 matrix $\mathbf{a}$. Furthermore, there exists a 1×2 vector $\mathbf{a}' = [1\ \ p]$ satisfying the equation

$$\mathbf{a}'\mathbf{A} = \mathbf{0}. \tag{16}$$

Both Eqs. (13) and (16) are equivalent and correspond to the equilibrium equations.

4. The branching problem

Following Lazopoulos (submitted for publication) and Lazopoulos and Markatis (1994), branching of the equilibrium problem exists if there exists an injective linear map $\mathbf{a_1} : R^1 \to R^5$ and a symmetric bilinear map $\mathbf{b} : R^1 \to L(R^5,R^6)$ satisfying the equation

$$(\mathbf{Ba}^2 + \mathbf{Ab})\mathbf{a_1} = \mathbf{0}, \tag{17}$$

where $\mathbf{B} = \frac{\partial \mathbf{A}}{\partial \mathbf{v}} = \begin{bmatrix} \left[\frac{\partial^2 V}{\partial v_k \partial v_l}\right] \\ \left[\frac{\partial^2 J}{\partial v_k \partial v_l}\right] \end{bmatrix}$.

Further, multiplying Eq. (17) by the co-kernel $\mathbf{a}'$ and recalling the equilibrium Eq. (16) we get the simplified version:

$$\mathbf{L}[\mathbf{dx}] = \mathbf{L}[\mathbf{aa}_1] = \mathbf{a}'\mathbf{Ba}^2\mathbf{a}_1 = 0. \tag{18}$$

In this case the incremental $\mathbf{dv}$ equals,

$$\mathbf{dv} = \xi\mathbf{aa_1} + \mathrm{o}(\xi), \tag{19}$$

where ξ is a parameter.

As it has been pointed out (Lazopoulos, in press), when the operator $\mathbf{L}$ is singular, branching of the equilibrium path takes place. Of course the transitions are classified as second order according to Landau et al. (1980). In this case no two-phase deformations are allowed. Nevertheless, the strain jumping compatibility condition is embedded (James, 1981; Gurtin, 1983)

$$[\mathbf{F}]\mathbf{f} = (\mathbf{F}^+ - \mathbf{F}^-)\mathbf{f} = \mathbf{0}, \tag{20}$$

where **F** denotes the deformation gradient, [ ] the jumping and **f** two (unit) vectors defining the plane of the phase boundary. Hence the gradient of deformation for the piece-wise homogeneous deformation is expressed by

$$\mathbf{F}^{\pm} = \mathbf{F}_0 + \xi^{\pm}\mathbf{F}_1, \tag{21}$$

where ξ^{+} and ξ^{-} are ξ parameters of Eq. (19), defined by the higher order terms of the governing equilibrium equation and the subindex zero denotes the large equilibrium placement. Furthermore, since the incremental deformation gradient $\mathbf{F}_1$ depends completely on the kernel **dx** of the operator **L,** defined by Eq. (18), see Lazopoulos (in press)

$$\mathbf{F}_1 = \begin{vmatrix} \mathrm{d}x_1 & \mathrm{d}x_4 & \mathrm{d}x_5 \\ \mathrm{d}y_1 & \mathrm{d}x_2 & \mathrm{d}x_6 \\ \mathrm{d}y_2 & \mathrm{d}y_3 & \mathrm{d}x_3 \end{vmatrix}, \tag{22}$$

where $\mathrm{d}y_i$, $i = 1,2,3$ are linear combinations of $\mathrm{d}x_j$, $j = 1,\ldots,6$ solving the equations of the conservation of the rotational momentum. Thus, the deformation jumping condition, Eq. (20), requires for non-zero **f** (direction of the phase boundary)

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}y_1} &= \frac{\mathrm{d}x_4}{\mathrm{d}x_2} = \frac{\mathrm{d}x_5}{\mathrm{d}x_6}, \\ \frac{\mathrm{d}x_1}{\mathrm{d}y_2} &= \frac{\mathrm{d}x_4}{\mathrm{d}y_3} = \frac{\mathrm{d}x_5}{\mathrm{d}x_3}. \end{aligned} \tag{23}$$

Eqs. (23) are the jumping compatibility conditions expressed exclusively by the components of the **L** operator. Consequently, the critical condition, interpreted into the existence of a kernel **dx** of the operator **L,** satisfying further the deformation gradient jumping conditions, is a necessary but not sufficient condition for the two-phase deformation. Following the procedure corresponding to the one-dimensional case, two-phase deformations will be developed in the neighborhood of the cusp singularity, because it is the lowest order cuspoid including Maxwell's sets, required for globally stable transitions. Therefore the conditions for the existence of the cusp singularity will be obtained under the action of the strain jump compatibility conditions, Eqs. (23).

Proceeding to the analysis (Lazopoulos, submitted for publication; Lazopoulos and Markatis, 1994), the cusp singularity appears when

$$\mathbf{a}'\mathbf{C}(\mathbf{a}\mathbf{a}_1)^3 + 3\mathbf{B}\mathbf{a}'(\mathbf{a}\mathbf{a}_1)(\mathbf{b}\mathbf{a}_1^2) = \mathbf{0}, \tag{24}$$

where $\mathbf{C} = \dfrac{\partial \mathbf{B}}{\partial \mathbf{v}}$.

Eq. (24) is equivalent to the existence of a multilinear function, $c \varepsilon L(R^1, L(R^1, L(R^5, R^6)))$ satisfying the equation,

$$\mathbf{C}(\mathbf{a}\mathbf{a}_1)^3 + 3\mathbf{B}(\mathbf{a}\mathbf{a}_1)(\mathbf{b}\mathbf{a}_1^2) + \mathbf{A}\mathbf{c}\mathbf{a}_1^3 = \mathbf{0}. \tag{25}$$

Likewise, Eq. (24) is equivalent to the existence of a vector $\hat{\mathbf{a}}_1 : R^1 \rightarrow R^5$ satisfying the equation

$$\mathbf{C}(\mathbf{a}\mathbf{a}_1)^2\mathbf{a} + 2\mathbf{B}(\mathbf{a}\mathbf{a}_1)(\mathbf{b}_1\mathbf{a}_1) + \mathbf{B}(\mathbf{b}\mathbf{a}_1)^2\mathbf{a} + \mathbf{A}\mathbf{c}\mathbf{a}_1^2 + (\mathbf{B}\mathbf{a}^2 + \mathbf{A}\mathbf{b})\hat{\mathbf{a}}_1 = \mathbf{0}. \tag{26}$$

The critical points satisfying Eq. (17) but failing to satisfy the Eq. (24) are classified according to Thom's (Gibson, 1979) classification theorem as fold points.

Thus the multilinear function **c** is defined by Eq. (25). Furthermore, the vector $\hat{\mathbf{a}}_1$ is respectively defined by Eq. (26). Likewise, the relation

$$\Theta = \mathbf{a}'\{\mathbf{D}(\mathbf{aa_1})^4 + 6\mathbf{C}(\mathbf{aa_1})^2(\mathbf{ba_1^2}) + \mathbf{B}(\mathbf{ca_1^3})(\mathbf{aa_1}) + 2\mathbf{C}(\mathbf{aa_1})^2(\mathbf{a\hat{a}_1}) + 2\mathbf{B}((\mathbf{ba_1^2})^2 + (\mathbf{ba_1^2})(\mathbf{ba_1\hat{a}_1}) + (\mathbf{a_j a_1})(\mathbf{ca_1^3} + \mathbf{ba_1\hat{a}_1})) + \mathbf{C}(\mathbf{a\hat{a}_1})(\mathbf{aa_1})^2 + \mathbf{B}(\mathbf{a\hat{a}_1})(\mathbf{ba_1^2}) + \mathbf{B}(\mathbf{ba_1\hat{a}_1})(\mathbf{aa_1}) + \mathbf{B}(\mathbf{aa_1})(\mathbf{ba_1\hat{a}_1})\} \neq \mathbf{0} \tag{27}$$

assures the singularity is not higher than the cusp. In addition, the total potential energy density in the neighborhood of the cusp singularity is given by

$$\widehat{V} = V + p(J-1) = \frac{\Theta}{4!}\xi^4 + \frac{\xi^2}{2}(\dot{\widehat{V}}_{ik}\,\mathrm{d}x_i\,\mathrm{d}x_k\,\mathbf{dt} + \widehat{V}'_{ik}\,\mathrm{d}x\,\mathrm{d}x\,\mathbf{d\mu}) + \xi(\dot{\widehat{V}}_i\,\mathrm{d}x_i\,\mathbf{dt} + \widehat{V}'_i\,\mathrm{d}x_i\,\mathbf{d\mu}). \tag{28}$$

Since the simplest (lowest) singularity including Maxwell's sets is the cusp catastrophe (Gilmore, 1981), the present discussion is limited to that singularity. Higher singularities include anyway Maxwell's sets and may be useful for many other problems with many (more than two) global minima. Indeed, the total potential energy V is expressed in this case by

$$\widehat{V} = s^4 + as^2 + bs. \tag{29}$$

Just exploring the control space (a,b) of the cusp singularity, see Fig. 1, the stable regions A_1 with unique minimum of the total energy function are prescribed by the fold curve A_2 where multiple local extremals are shown up. Further, the fold curve is described by the relation

$$8a^3 + 27b^2 = 0. \tag{30}$$

That is, also, the critical curve for locally stable transitions. Nevertheless, the set for globally stable transitions, called Maxwell's set, is the semi-axis in the control space with $a < 0$ (see Fig. 1). In that case equilibrium states with

$$\xi = \pm\sqrt{-\frac{2a}{3}} \tag{31}$$

yield global minima of the total energy function. Hence the deformation gradient

$$\mathbf{F} = \mathbf{F^0} + \xi\mathbf{F^1} = \mathbf{F^0} \pm \sqrt{-\frac{2a}{3}}\mathbf{F^1} \tag{32}$$

suffers jumping at a plane defined by the vectors $\mathbf{f}$, see Eq. (20). Furthermore, the Piola–Kirchhoff stress is defined by

$$\mathbf{T}(\mathbf{F}) = \frac{\partial(W + p(\det\mathbf{F} - 1))}{\partial\mathbf{F}}. \tag{33}$$

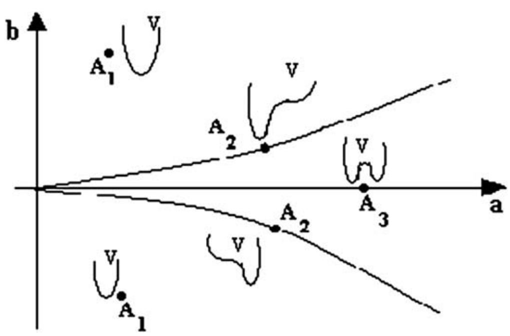

Fig. 1. Geometry of the cusp control space.

Hence,

$$\mathbf{T}(\mathbf{F}^{\pm}) = \left.\frac{\partial(W + p(\det\mathbf{F} - 1))}{\partial\mathbf{F}}\right|_{\xi=0} + \left.\frac{\partial^2(W + p(\det\mathbf{F} - 1))}{\partial\mathbf{F}^2}\right|_{\xi=0} \cdot \xi^{\pm}\mathbf{F_1} + \frac{1}{2}\left.\frac{\partial^3(W + p(\det\mathbf{F} - 1))}{\partial\mathbf{F}^3}\right|_{\xi=0} \xi^2\mathbf{F_1^2} + \mathrm{o}(\xi^2). \tag{34}$$

Since

$$\mathbf{a}'\mathbf{Badx} = \left.\frac{\partial^2(W + p(\det\mathbf{F} - 1))}{\partial\mathbf{F}^2}\right|_{\xi=0} \cdot \mathbf{F_1} = 0. \tag{35}$$

Eq. (34) reveals,

$$T^+ = T^- + \mathrm{o}(\xi^2). \tag{36}$$

Therefore, Eq. (36) covers the equilibrium requirement of the same stress vector at the two phases of the phase boundary. In addition, the total potential energy density function is the same at both phases, because on Maxwell's sets

$$V^+ = V^-. \tag{37}$$

Consequently, Maxwell's condition see Gurtin (1983)

$$(W + p(\det\mathbf{F} - 1))^+ - (W + p(\det\mathbf{F} - 1))^- = \mathbf{T}^{\pm}(\mathbf{F}^+ - \mathbf{F}^-) \tag{38}$$

is revealed.

5. Application

The present application of simple plane shear implements the theory exposed in the preceding sections. Although the theory has not imposed any restrictions on the anisotropy of the homogeneous material, the chosen incompressible material is a specific isotropic material with strain energy density function

$$W(I_1) = aI_1 + b \cdot I_1^{3/2} + c, \tag{39}$$

where I_1 is the first strain invariant. It is evident that the parameters a, b, c have dimensions of energy density. Just to avoid the discussion for the dimensions of the various parameters and various quantities the problem is considered already dimensionless. Non-dimensionalization has already been performed. We consider homogeneous deformations with displacements components $u_{ij} = 0$ if i or j are equal to 3. In that case

$$I_1 = (1 + u_{11})^2 + u_{21}^2 + u_{12}^2 + (1 + u_{22})^2, \tag{40}$$

$$J = \det\mathbf{F} = 1 + u_{11} + u_{22} + u_{11}u_{22} - u_{12}u_{21} = 1. \tag{41}$$

Since $W(I_1)$ has to satisfy zero values and zero stresses at the reference placement, the following relations are valid, see Knowles and Sternberg (1978).

$$W|_{(u_{11}=0,u_{22}=0)} = 0, \quad \left.\frac{\partial W}{\partial u_{11}}\right|_{(u_{11}=0,u_{22}=0)} = 0. \tag{42}$$

Hence the function W, satisfying Eqs. (42), should be

$$W = aI_1 + \frac{\sqrt{2}}{3} aI_1^{3/2} - \frac{2}{3}\alpha. \tag{43}$$

The problem of the initial simple shear in the x_1 direction will be discussed. The emergence of discontinuous deformation gradients will be exhibited and the piece-wise constant strain field will be described. Since the simple shear is incompressible

$$\begin{aligned} &u_{11} = u_{21} = u_{22} = 0, \\ &u_{12} = k \end{aligned} \tag{44}$$

prescribes the pre-critical deformation with the corresponding first Piola–Kirchhoff stress tensor elements, $t_{11}, t_{22}, t_{12}, t_{21}$.

Furthermore, the four strain components are not independent. The equation expressing the conservation of the rotational momentum, see Lazopoulos (in press)

$$u_{11} = -1 + (t_{11}u_{21} + t_{12}(1 + u_{22}) - t_{22}u_{12})/t_{21}. \tag{45}$$

Let us recall the density of the total potential energy is equal to

$$V = W - t_{11}u_{11} - t_{12}u_{12} - t_{21}u_{21} - t_{22}u_{22}. \tag{46}$$

In fact the parameters controlling the problem are the shearing k and the modulus a of the strain energy density function. Those two parameters define the emergence of discontinuous strain fields.

Since the problem of the initial simple shear is discussed, we try to locate the critical point where the operator **L** becomes singular. Recalling Eq. (45), the total energy density V depends on the three strain components u_{11}, u_{21}, u_{22}, i.e.

$$V = V(u_{12}, u_{21}, u_{22}). \tag{47}$$

Considering at the critical point

$$\begin{aligned} &u_{12} = k + \xi x_1, \\ &u_{21} = \xi x_2, \\ &u_{22} = \xi x_3 \end{aligned} \tag{48}$$

the total potential energy density $V(u_{12}, u_{21}, u_{22})$ is expanded around the strains $(u_{12}, u_{21}, u_{22}) = (k, 0, 0\ )$. Implementing the proposed theory we derive the operator

$$\mathbf{A} = \begin{bmatrix} \dfrac{\partial V}{\partial u_{12}} & \dfrac{\partial V}{\partial u_{21}} & \dfrac{\partial V}{\partial u_{22}} \\ \dfrac{\partial J}{\partial u_{12}} & \dfrac{\partial J}{\partial u_{21}} & \dfrac{\partial J}{\partial u_{22}} \end{bmatrix} \quad \text{at } (u_{12} = k,\ u_{21} = 0,\ u_{22} = 0). \tag{49}$$

The space of the solutions **a** satisfying the equilibrium equation $\mathbf{Aa} = \mathbf{0}$ is defined by

$$\mathbf{a} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ x & y \end{bmatrix} \tag{50}$$

with

$$x = -\frac{\partial J}{\partial u_{21}} \bigg/ \frac{\partial J}{\partial u_{22}} = -\frac{S_{11}}{2S_{12}} \quad \text{and} \quad y = -\frac{\partial J}{\partial u_{12}} \bigg/ \frac{\partial J}{\partial u_{22}} = \frac{S_{22}}{2S_{12}} \quad \text{at } (u_{12} = k, u_{21} = 0, u_{22} = 0). \tag{51}$$

Moreover, the equilibrium equation $\mathbf{a}'\mathbf{A} = \mathbf{0}$ with the cokernel $\mathbf{a}' = [1, p]$ yields at $(u_{12} = k, u_{21} = 0, u_{22} = 0)$.

$$A_1 = \frac{\partial V}{\partial u_{12}} + p\frac{\partial J}{\partial u_{12}} = 0, \tag{52a}$$

$$A_2 = \frac{\partial V}{\partial u_{21}} + p\frac{\partial J}{\partial u_{21}} = 0, \tag{52b}$$

$$A_3 = \frac{\partial V}{\partial u_{22}} + p\frac{\partial J}{\partial u_{22}} = 0. \tag{52c}$$

The critical strains are defined when the operator $\mathbf{L}$, see Eq. (18), is singular. The operator $\mathbf{B} = \nabla\mathbf{A}$ is defined by

$$B = \begin{bmatrix} \begin{bmatrix} \frac{\partial^2 A}{\partial u_{12}^2} & \frac{\partial^2 A}{\partial u_{12}\partial u_{21}} & \frac{\partial^2 A}{\partial u_{12}\partial u_{22}} \\ \frac{\partial^2 A}{\partial u_{21}\partial u_{12}} & \frac{\partial^2 A}{\partial u_{21}^2} & \frac{\partial^2 A}{\partial u_{21}\partial u_{22}} \\ \frac{\partial^2 A}{\partial u_{22}\partial u_{12}} & \frac{\partial^2 A}{\partial u_{22}\partial u_{21}} & \frac{\partial^2 A}{\partial u_{22}^2} \end{bmatrix} \\ \begin{bmatrix} \frac{\partial^2 J}{\partial u_{12}^2} & \frac{\partial^2 J}{\partial u_{12}\partial u_{21}} & \frac{\partial^2 J}{\partial u_{12}\partial u_{22}} \\ \frac{\partial^2 J}{\partial u_{21}\partial u_{12}} & \frac{\partial^2 J}{\partial u_{21}^2} & \frac{\partial^2 J}{\partial u_{21}\partial u_{22}} \\ \frac{\partial^2 J}{\partial u_{22}\partial u_{12}} & \frac{\partial^2 J}{\partial u_{22}\partial u_{21}} & \frac{\partial^2 J}{\partial u_{22}^2} \end{bmatrix} \end{bmatrix} \quad \text{at } (u_{12} = k, u_{21} = 0, u_{22} = 0). \tag{53}$$

Furthermore, the cokernel $\mathbf{a}' = [1, p]$ and the vector $\mathbf{a_1} = \begin{bmatrix} 1 \\ z \end{bmatrix}$ yields the two-dimensional vector

$$\mathbf{a}'\mathbf{Ba}^2\mathbf{a_1} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{54}$$

Recalling Eq. (17), the multilinear operator $\mathbf{b}$ is defined by the equation

$$(\mathbf{Ba}^2 + \mathbf{Ab})\mathbf{a_1} = \mathbf{0} \tag{55}$$

with

$$\mathbf{b} = \begin{bmatrix} \begin{bmatrix} b1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} b2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}. \tag{56}$$

Then the cusp condition, Eq. (24) is computed. Hence,

$$\text{Cusp} = \mathbf{a}'(\mathbf{C}(\mathbf{aa_1})^3 + 3\mathbf{B}(\mathbf{aa_1})(\mathbf{b_1a_1^2})) = \mathbf{0}. \tag{57}$$

Likewise, the involved deformation gradient jumping condition, Eq. (23), is expressed in the present case by

$$u_{11}u_{22} - (u_{12} - k)u_{21} = 0. \tag{58}$$

Recalling Eq. (45) and expanding the strains around the initial simple shear deformation (see Eqs. (48)), (58) is reduced to

$$\text{Comp} = -x_1x_2 + \frac{x_3(-S_{22}x_1 + S_{11}x_2 + S_{12}x_3)}{S_{21}} = 0 \tag{59}$$

with,

$$x_1 = z, \quad x_2 = 1, \quad x_3 = x + zy. \tag{60}$$

Let us point out that for the emergence of the strain jumping conditions (52a), (52b), (52c), (54), (57), and (59) have to be simultaneously satisfied.

With the help of the Mathematica computerized algebra pack (Wolfram, 1996), the set of the critical parameters $(a^0, p^0, k^0, z^0, S_{11}^0, S_{22}^0, S_{12}^0, S_{21}^0)$ satisfying conditions (52a), (52b), (52c), (54), (57), and (59), is defined. Indeed, a critical state satisfying the cusp condition and compatible with strain jumping is defined by the set

$$\begin{pmatrix} a^0 = 0.880, & p^0 = -0.150, & k^0 = 0.830, & z^0 = 0.970, \\ S_{11}^0 = -0.029, & S_{22}^0 = 0.184, & S_{12}^0 = -0.028, & S_{21}^0 = 0.181 \end{pmatrix}. \tag{61}$$

It is recall that $\mathbf{a}' = [1, p^0]$ and the vector $\mathbf{a_1} = \begin{bmatrix} 1 \\ z^0 \end{bmatrix}$ are defined by the critical values. Likewise, the matrix $\mathbf{a}$ is defined by Eqs. (51) and (52) with

$$x = -\frac{S_{11}}{2S_{12}} = -0.52, \quad y = \frac{S_{22}}{2S_{12}} = -3.28. \tag{62}$$

Indeed,

$$\mathbf{a} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ -0.52 & -3.28 \end{bmatrix}. \tag{63}$$

Therefore,

$$x_1 = z = 0.97, \quad x_2 = 1, \quad x_3 = x + zy = 2.66. \tag{64}$$

The elements b_1, b_2 of the matrix $\mathbf{b}$, are defined through Eqs. (54) and (55) and are equal to

$$b_1 = -10.665, \quad b_2 = 12.954. \tag{65}$$

The next step is the definition of the matrix $\mathbf{c}$ through satisfying Eq. (25). Indeed,

$$\mathbf{c} = \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} -37.10 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} & \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \end{bmatrix}. \tag{66}$$

In addition, Eq. (26) yields the vector $\hat{\mathbf{a}}_1 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. In fact Eq. (26) results in the two equations

$$\begin{aligned} &-0.81v_1 + 0.86v_2 - 18.43 = 0, \\ &0.88v_1 - 0.88v_2 + 20.43 = 0 \end{aligned} \tag{67}$$

with solution,

$$v_1 = -28.23, \quad v_2 = -5.40. \tag{68}$$

Now Θ, see Eqs. (27), may be defined and it is found equal to

$$\Theta = 15467.9. \tag{69}$$

Let us, further, consider incremental values of the controlling parameters, a and k, i.e.

$$\begin{aligned} a &= a^0 + \delta a, \\ k &= k^0 + \delta k. \end{aligned} \tag{70}$$

The function $\widehat{V}$, see Eq. (28), in the present case is given by

$$\widehat{V} = 644.45\xi^4 - (2.44\delta a + 3.52\delta k)\xi^2 + (0.74\delta a + 2.93\delta k)\xi = 0. \tag{71}$$

Since Maxwell's set is described when the linear for ξ term is zero

$$\delta k = -0.252\delta a. \tag{72}$$

Hence the energy function $\widehat{V}$ on the Maxwell set is given by

$$\widehat{V} = 644.45\xi^4 - 1.554\delta a\xi^2. \tag{73}$$

The equilibrium equation yields

$$2.577.8\xi^3 - 3.107\delta a\xi = 0 \tag{74}$$

with

$$\xi = \pm 0.035\sqrt{\delta a} \quad \text{and} \quad \xi = 0. \tag{75}$$

Recalling the strain jumping compatibility Eq. (58), the incremental gradient deformation $\xi\mathbf{F_1}$, Eq. (35), is defined by

$$\xi\mathbf{F}_1 = \xi\begin{bmatrix} x_1x_2/x_3 & x_1 \\ x_2 & x_3 \end{bmatrix} = \xi\begin{bmatrix} 0.365 & 0.97 \\ 1 & 2.66 \end{bmatrix}. \tag{76}$$

Hence,

$$\begin{aligned} u_{11} &= 0.365\xi, \\ u_{12} &= 0.83 + 0.97\xi, \\ u_{21} &= \xi, \\ u_{22} &= 2.66\xi, \end{aligned} \tag{77}$$

where ξ is defined by Eq. (75). Besides, the direction $\mathbf{f}$ of the phase boundary may be defined by the equation

$$\mathbf{F_1 f} = \mathbf{0}. \tag{78}$$

In the present case, the unit vector $\mathbf{f}$ directed parallel to the phase boundary is found to be equal to

$$\mathbf{f} = (-0.35, \quad 0.94)^{\mathrm{T}}. \tag{79}$$

Recalling Eq. (34) which gives the first Piola–Kirchhoff stress tensor and taking into consideration Eq. (35), the components of the first Piola–Kirchhoff stress tenor may be evaluated by

$$\begin{aligned} S_{11} &= -0.029 + 2.463\xi^2, \\ S_{12} &= -0.028 + 3.983\xi^2, \\ S_{21} &= 0.181 + 2.907\xi^2, \\ S_{22} &= 0.184 + 9.130\xi^2. \end{aligned} \tag{80}$$

Let me recall that the controlling the problem parameters are the modulus a of the stain energy density and the shearing k. The increments of those parameters should be related through Eq. (72). Therefore Eq. (75) yields the relation

$$\xi = \pm 0.035\sqrt{\delta a} = \pm 0.111\sqrt{-\delta k}. \tag{81}$$

It is evident that the increment δa of the modulus a must be positive, whereas the increment δk of the shearing k must be negative.

Therefore, the discontinuous deformation gradient strain field has completely been defined.

6. Conclusion

It has been found that bifurcation is a necessary condition for emergence of discontinuous strains in (piece-wise) homogeneous deformations. Nevertheless it is not sufficient. The deformation gradient jumping compatibility condition restricts the kernel space of the branching problem. Furthermore, globally stable transitions, requiring multiple global minima, are shown up if the cusp condition for the total potential energy density function is satisfied. In fact the existence of Maxwell's set, allowing for multiple global minima, require at least the cusp condition. Consequently the branching critical condition should be combined with the strain jumping and cusp conditions for the emergence of discontinuous strain fields. That approach is applied to incompressible materials in the present work. The procedure is more complicated than in non-constrained materials. The various steps have been explained in the application implementing the theory. Three-dimensional sectionally homogeneous problems in any incompressible anisotropic material may be studied applying the present analysis employing singularity theory. Following the present procedure, the emergence of discontinuous strain fields may be studied for any constrained material, even with multiple constraints.

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